Thm (Ostrowski): Up to equivalence, the non-trivial norms on $\mathbb{Q}$ are

$|-|_p$, p prime & $|-|_\infty$

p-adic norm — usual absolute norm

completions $\mathbb{Q}_p$ — $\mathbb{R}$

Def: K number field. A place of K is an equivalence class of (non-trivial) norms on K

$\mathcal{P}(K) :=$ set of places of K

Ex: 1) $\tau : K \hookrightarrow \mathbb{C}$

$\rightsquigarrow$ $|-|_\tau = |-|_{\mathbb{C}} \circ \tau$ is a norm on K

$\tau, \tau' : K \hookrightarrow \mathbb{C}$

$|-|_\tau$ equivalent to $|-|_{\tau'}$

$\Leftrightarrow$ $\overline{\tau} = \tau'$ or $\tau = \tau'$

"archimedean or infinite places"

2) $\mathfrak{p} \subseteq \mathcal{O}_K$ max'l

"non-arch. or finite

$\Rightarrow \quad |-|_{\mathfrak{p}} := N(\mathfrak{p})^{-v_{\mathfrak{p}}(-)}$ "places"

$v_{\mathfrak{p}} : K \to \mathbb{Z} \cup \{\infty\}$ $\mathfrak{p}$-adic valuation

$|-|_{\mathfrak{p}}$ equiv. to $|-|_{\mathfrak{q}}$ iff $\mathfrak{p} = \mathfrak{q}$

Thm: These constitute all places of K

Let L/K be a finite ext. of numberfields,
write $w|v$ if w restricts to v on K
$w \in \mathcal{O}(L)$, $v \in \mathcal{O}(K)$ (More prec. a repr. in w restricts to a repr. on v)

Notation: $v \in \mathcal{O}(K) \rightsquigarrow K_v$ compl. for v

$w \in \mathcal{O}(L) \rightsquigarrow L_w$

Thm: 1) $$L \otimes_K K_v \simeq \prod_{\substack{w|v \\ w \in \mathcal{O}(L)}} L_w$$

(analog of $K \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$)

2) If $v \in \mathcal{O}(K)$ non-arch, with ass. prime $\mathfrak{p} \subseteq \mathcal{O}_K$, $w \in \mathcal{O}(L)$, $w | v$ with ass. prime $\mathfrak{q} \subseteq \mathcal{O}_L$, then

$$f(L_w | K_v) = f(\mathfrak{q} | \mathfrak{p}) = [k(\mathfrak{q}) : k(\mathfrak{p})]$$

$$e(L_w | K_v) = e(\mathfrak{q} | \mathfrak{p})$$

3) If L/K Galois, then L_w/K_v Galois

& $\mathrm{Gal}(L_w/K_v) = D(\mathfrak{q} | \mathfrak{p})$

$$D(\mathfrak{q}|\mathfrak{p}) = \{ \sigma \in \mathrm{Gal}(L/K) \mid \sigma(\mathfrak{q}) = \mathfrak{q} \} = \{ \sigma \in \mathrm{Gal}(L/K) \mid \sigma(w) = w \}$$

$w \in \mathcal{O}(L)$

$w = [|-|']$, $|-|' : L \to \mathbb{R}_{\geq 0}$ norm

$\Rightarrow$ $\sigma(w) = [|-|' \circ \sigma]$ for $\sigma \in \mathrm{Gal}(L/K)$

Prf: 1) Write $L = K[x]/f(x)$

$$f(x) = \prod_{i=1}^{r} g_i(x) \in K_v[x]$$

g_i irred., pairwise coprime

$$\Rightarrow L \otimes_K K_v \simeq \prod_{i=1}^{r} L_i$$

$L_i = K_v[x]/(g_i(x))$

L_i: completion of L at $|\cdot|_{L_i}$

$|\cdot|_{L_i}$: unique ext. of $|\cdot|_{K_v}$ to L_i

$$\Rightarrow L \otimes_K K_v \hookrightarrow \prod_{\substack{w \mid v \\ w \in \sigma(\cancel{L})}} L_w$$

but $L \otimes_K K_v \twoheadrightarrow L_w$ for all $w \mid v$

(image dense + closed by completeness of f.d. K_v-v.s.)

$\Rightarrow L_w \simeq L_i$ for some i

2) Exercise

3) follows from 2) as ex. nat. morph.

$$D(\mathfrak{q}|\mathfrak{p}) \hookrightarrow Gal(L_w/K_v)$$

$$\sigma \mapsto \text{ext. of } \sigma \text{ to } L_w \to L_w$$

$$\begin{array}{c} L_w \\ \uparrow \quad \nearrow \iota\circ\sigma \\ L \end{array}$$ cont. for the w-top

$\iota : L \hookrightarrow L_w$

Inj. as $L \subseteq L_w$ dense

$\square$

Thm (Hermite-Minkowski)

let K be a number field, $S \subseteq \mathcal{O}(K)$ finite containing the infinite places, $n \geq 0$

$\Rightarrow \{L/K$ unramified outside S, $[L:K] \leq n\}/_{\text{isom.}}$ is finite

$\Updownarrow$

$\forall v \in \mathcal{O}(K) \setminus S$
L_w/K_v unramified
$\forall w \in \mathcal{O}(L), w | v$

($\pi_1^{ét}(\operatorname{Spec} \mathcal{O}_{K,S}, \bar{x})$ has only fin. many quot. of order $\leq n$)

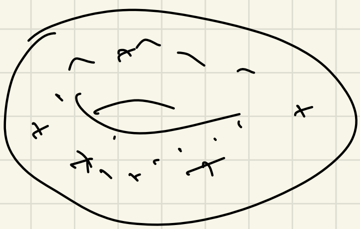
$\mathcal{O}_{K,S}$ localisation with

$$\operatorname{Spec} \mathcal{O}_{K,S} = \operatorname{Spec} \mathcal{O}_K \setminus \{ v \in \sigma_f(K) \mid n \}$$

$\sigma_f(K)$ finite places

$$\pi_1^{ét}(\operatorname{Spec} \mathcal{O}_{K,S}, \bar{x}) = Gal(K_S/K),$$

K_S = max'l ext. of K, which is unramified outside S



Proof (Sketch): Wlog $K = \mathbb{Q}$

Fix $\Delta \in \mathbb{Z}$

Recall: $\# \{ L/\mathbb{Q}, \Delta_L = \Delta \} < \infty$

$\Rightarrow$ Suff. to bdd Δ_L in terms of S, n

$$\text{But } (\Delta_L) = \underset{\substack{\uparrow \\ \text{different}}}{N_{L/\mathbb{Q}}(\delta_{L/\mathbb{Q}})} = \prod_{\substack{p \in S \\ v \mid p \\ v \in \mathcal{O}(L)}} N_{L_v/\mathbb{Q}_p}(\delta_{L_v/\mathbb{Q}_p})$$

$$L \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{\substack{v \in \mathcal{O}(L) \\ v \mid p}} L_v$$

and $\mathbb{Q}_p$ has only fin. many ext. of degree $\leq n$

($\leadsto$ p-adic valuations of the differents of ext's is bdd in terms of n)

$\Rightarrow$ Δ_L is bdd as desired $\square$

Class field theory ($\equiv$ theory of abelian ext's of number fields)

For $\mathbb{Q}$:

$$\underset{\substack{\uparrow \\ \text{canonical isom.}}}{Gal(\mathbb{Q}^{ab}/\mathbb{Q}) \simeq} \hat{\mathbb{Z}}^\times \quad \text{as } \mathbb{Q}^{ab} = \bigcup_N \mathbb{Q}(\mu_N)$$

(Kronecker-Weber)

[$\Rightarrow$ arithmetic consequences:

$K/\mathbb{Q}$ is finite abelian, $K \subseteq \mathbb{Q}(\mu_N)$

$\Rightarrow$ understand the decom. of primes $p \nmid N$ in K via congruences mod N by identifying the Frobenius elements]

$$\sigma_p \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \overset{1:1}{\longleftrightarrow} p \in (\mathbb{Z}/N)^\times$$

Q: How to generalize this to arbitrary number fields?

$K/\mathbb{Q}$ finite $\rightsquigarrow$ Can we describe K^{ab}? Unknown in general

Or $\mathrm{Gal}(K^{ab}/K)$? Yes!

Need: Idèle class group

$K/\mathbb{Q}$ finite, $\sigma(K) = \sigma_f(K) \amalg \sigma_\infty(K)$

finite places of K infinite places of K

Def: Ring of adèles

$$\mathbb{A}_K := \left\{ (x_v)_{v \in \mathcal{O}(K)} \in \prod_{v \in \mathcal{O}(K)} K_v \;\middle|\; |x_v|_v \leq 1 \text{ for almost all } v \right\}$$

$$= \mathbb{A}_{K,f} \times \mathbb{A}_{K,\infty}$$

$$\mathbb{A}_{K,\infty} = K \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v \in \mathcal{O}_\infty(K)} K_v$$

Spec $\mathbb{A}_K$ complicated, similar to Spec $\prod_{\mathbb{N}} k$

$$\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{Q} = 0$$

E.g. $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \big(\underbrace{\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}}_{\prod_p \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q}}\big)$

$$\operatorname{Spec} \mathbb{A}_{\mathbb{Q}} \supseteq \operatorname{Spec}\big(\underbrace{(\prod_p \mathbb{F}_p) \otimes_{\mathbb{Z}} \mathbb{Q}}_{\neq 0}\big)$$

Note: the diagonal embedding

$$K \hookrightarrow \prod_{v \in \mathcal{O}(K)} K_v$$ lands inside $\mathbb{A}_K$

Def: $\mathbb{A}_K^* =$ units in $\mathbb{A}_K$ "idèle group"

$$= \{ (x_v)_{v \in \mathcal{O}(K)} \in \prod_{v \in \mathcal{O}(K)} K_v^* \mid |x_v|_v = 1$$ for all but fin. many places $v \}$

Again: $K^\times \overset{\iota}{\hookrightarrow} \prod_{v \in \mathcal{O}(K)} K_v^*$ (diagonal)

lands inside $\mathbb{A}_K^\times$, $\iota(K^\times) =$ grp. principal idèles

Def: $\mathbb{A}_K^* / \iota(K^\times)$ idèle class group

Let's analyze the case $K = \mathbb{Q}$

Le: $\mathbb{A}^\times / \mathbb{Q}^\times \simeq \hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$

Prf: Let $(x_v)_v \in \mathbb{A}_K^* \Rightarrow |x_v|_v = 1$ for almost all $v \in \mathcal{O}(\mathbb{Q})$

$\Rightarrow$ After multiplying by some elt in $\mathbb{Q}$, we may assume that

$|x_v|_v = 1$ for all $v \in \mathcal{O}_f(K)$

& $x_v > 0$ for v arch.

$$\Rightarrow \quad (x_v)_v \in \quad (\hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0})$$

But $\hat{\mathbb{Z}}^\times \cap \mathbb{R}_{>0} \cap \iota(\mathbb{Q}^\times)$

$$= \{x \in \mathbb{Q}^\times \mid |x|_v = 1 \ \forall v \in \mathcal{O}_f(\mathbb{Q}), \ x > 0\}$$

$$= \{x \in \mathbb{Z}^\times \mid x > 0\} = \{1\} \qquad \square$$

Upshot: $\exists$ can. cont. surj.

$$\mathbb{A}^\times / \mathbb{Q}^\times \xrightarrow{\sim} \hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$$

with kernel the conn. comp. of 1 in $\mathbb{A}^\times/\mathbb{Q}^\times$

Thm (Global class field theory) For $K/\mathbb{Q}$ finite $\exists$ can. cont. surj.

"Artin reciprocity"

$$\mathrm{rec}_K : \mathbb{A}^\times / K^\times \twoheadrightarrow \mathrm{Gal}(K^{ab}/K)$$

with kernel the conn. comp. of 1 in $\mathbb{A}^\times / K^\times$

(+ identification of Frob. in $\mathrm{Gal}(L/K)$, L/K finite abelian)